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A paper should contain a short and clear summary of the new results obtained and the relations in which they stand to results already known. Contributors are requested to bear in mind that, at the present stage of mathematical research, hardly any paper is likely to be so completely original as to be independent of earlier work in the same direction; and that readers are often helped to appreciate the importance of a new investigation by seeing its connection with earlier results.

The principal results of a paper should, when possible, be enunciated separately and explicitly in the form of definite theorems.

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OFFICE OF THE REGISTRAR OF BOOKS
24 FEB 1928
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M. T. NARANIENGAR, M.A.
Editor—1909—1927

Prof. M. T. Naraniengar who has been Editor of this Journal from its very start in 1909, has for some time past been desiring relief from active work, and the Committee felt they had no option but to yield to his wishes. The present Editors wish to take the opportunity afforded by his retirement to express their deep sense of the services rendered by Prof. Naraniengar towards the advancement of higher mathematical study and research in India. What the Journal is to-day is due in no small measure to the zealous care and unselfish attention which Prof. Naraniengar has showered on it during the past 19 years and in wishing him long life and happiness they feel that they are but voicing the sentiments felt by every member of the Society and every reader of this Journal.

where $0 < \gamma \leqslant \frac{\pi}{2}$, so that throughout the range of integration

$$\sin\theta \geqslant \frac{2}{\pi}\,\theta.$$

Therefore, if r is sufficiently large,

$$|\mathrm{I}| \leqslant \int_0^{\gamma} r \exp.\left\{-\mathrm{A}r\frac{2}{\pi}\,\theta - sd(\log r)^{\alpha}\cos\left(\frac{\alpha\gamma}{\log r}\right)\right\}d\theta,$$

$$= \exp.\left\{-sd(\log r)^{\alpha}\cos\left(\frac{\alpha\gamma}{\log r}\right)\right\}.r.\left(\frac{1-e^{-2\mathrm{A}r\gamma/\pi}}{2\mathrm{A}r/\pi}\right)$$

which $\rightarrow 0$, as $r\rightarrow\infty$, s being real, and greater than 0.

It follows that the integral (5.4) may be taken along the circle $|y| = 1$ (as intercepted by the vectors $\theta = 0$, and $\theta = \gamma$), and then along the radius vector $\theta = \gamma$, in the plane of the complex variable $y = re^{i\theta}$ instead of along the real axis. So that

$$(5.5)\qquad \mathrm{J}(s) = \int_1^{\infty}\exp.\left\{\mathrm{A}ire^{i\gamma} - sd(\log r + i\gamma)^{\alpha}\right\}e^{i\gamma}\,dr + \mathrm{Q}(s),$$

$\mathrm{Q}(s)$ being the contribution of the integral (5.4) taken along the circle

$$|y| = 1,\ \theta = 0,\ \theta = \gamma,$$

Now the second term on the right side of (5.5) obviously represents an integral function of s. Also the integral

$$\int_1^{\infty}\left|e^{i\gamma+\mathrm{A}ire^{i\gamma}-(\sigma+it)d((\log r)^2+\gamma^2)^{\alpha/2}(\cos\alpha\tau + i\sin\alpha\tau)}\right|dr,$$

$$\left(\tau = \tan^{-1}\frac{\gamma}{\log r}\right),$$

$$= \int_1^{\infty} e^{-\mathrm{A}r\sin\gamma - d((\log r)^2+\gamma^2)^{\alpha/2}(\sigma\cos\alpha\tau - t\sin\alpha\tau)}\,dr,$$

which is uniformly convergent throughout any finite domain of values of s. Hence the integral on the right side of (5.5) represents an integral function of s, and, therefore, $\mathrm{J}(s)$ is an integral function of s, and so is $\mathrm{H}(s)$.

Now suppose $k + 1 > \beta$, so that the series is divergent. Form

$$\mathrm{H}(s, z) = \sum_1^{\infty}\exp.\left\{\mathrm{A}in^{\beta} - s(\log n)^{\alpha}\right\}n^{-(z-1)},$$

the series being absolutely convergent for all bounded values of s, and

$$R(z) > 1 + k.$$

By Theorem IV, we have

$$H(s, z) = J(s, z) + G(s, z),$$

where $G(s, z)$ is an integral function of both s and z, and

$$J(s, z) = \int_1^{\infty} \exp. \left\{ Aix^{\beta} - s(\log x)^{\alpha} \right\} x^{-(z-k)}\, dx,$$

where $R(z) > 1 + k$, and s bounded.

Now, proceeding as above, it is easy to prove that the function $J(s, z)$ is an integral function of both s and z. So that $H(s, z)$ is an integral function of both s and z, and hence

$$\lim_{z \to 0} H(s, z)$$

is an integral function of s, which defines the divergent series (b).

That this should be so is also evident from the fact that the series (b) is summable (C)* all over the plane, for $k + 1 \geqslant \beta$.

(c) $\sum\limits_1^{\infty} \exp. \left\{ Ai(\log n)^{\gamma} - s(\log n)^{\alpha} \right\} n^k,\ A \neq 0,\ \gamma > 1,\ 0 < \alpha < 1.$

This series is convergent for all values of s, and so represents an integral function of s, if $k + 1 < 0$. If $k + 1 = 0$, it is conditionally convergent for $\sigma \geqslant 0$, but represents an integral function of s. When $k + 1 > 0$, the series is divergent.

Suppose, in the first place, that $k+1>0$, and form

$$H(s, z) = \sum_1^{\infty} \exp. \left\{ Ai(\log n)^{\gamma} - s(\log n)^{\alpha} \right\} n^{-(z-k)},$$

the series being absolutely convergent, if

$$R(z) > 1 + k > 0.$$

---

* G. H. Hardy: 'Theorems connected with Maclaurin's test for the convergence of series'; *Proc. Lond. Math. Soc.* Ser. 2, vol. 9, 1910.

Unlike the series

$$\sum_{1}^{\infty} \exp. \left\{ Ain^{\beta} - s(\log n)^{\alpha} \right\} n^{-(z-k)}$$

the present series is not summable (C)* all over the z-plane.

Applying Theorem IV, we have

$$H(s, z) = J(s, z) + G(s, z),$$

where $G(s, z)$ is an integral function of both s and z, and

$$J(s, z) = \int_{1}^{\infty} \exp. \left\{ Ai(\log x)^{\gamma} - s(\log x)^{\alpha} \right\} x^{-(z-k)} dx,$$

where $R(z) > 1 + k$, and s bounded.

Now proceeding, as in the case of integral (1.8), it is easy to prove that the function $J(s, z)$ is an integral function of both the variables s and z, so that $\lim_{z \to 0} H(s, z)$ is an integral function of s, which, therefore, defines the divergent series (c). For the case $k + 1 = 0$, apply Theorem III, instead of Theorem IV.

The last example naturally suggests the question whether or not such divergent Dirichlet's series as

$$\text{(i)} \quad \sum_{1}^{\infty} \exp. \left\{ Ain^{\gamma} + kn^{\beta} - sn^{\alpha} \right\},$$

$k > 0$, $1 \geqslant \gamma > \beta > \alpha > 0$, $A \neq 0$, nor a multiple of 2π, when $\gamma = 1$;

$$\text{(ii)} \quad \sum_{1}^{\infty} \exp. \left\{ -s(\log n)^{\alpha} + k(\log n)^{\beta} + Ai(\log n)^{\gamma} \right\},$$

$$k > 0, \quad A \neq 0, \quad \gamma > \beta > \alpha \geqslant 1,$$

none of which is summable (C), may be uniquely defined by properly generalizing our method of definition. That the answer is in the affirmative will be seen in Part II.

---

* G. H. Hardy: 'Example to illustrate a point in the theory of Dirichlet's series'; *The Tôhoku Mathematical Journal*, Vol. 8, No. 2, October 1915.

6. THEOREM V. *Let*

$$H(s) = \sum_{p}^{\infty} (-1)^n \phi(n) e^{-sf(n)},$$

where $\phi(x)$ and $f(x)$ satisfy the same conditions as in Theorem I; then the Dirichlet's series is convergent if $\sigma > k$, absolutely convergent if $\sigma > 1 + k$, and the function $H(s)$ is an integral function of s.

The conclusion as to the absolute convergence of the series follows from the fact that the nth term of the series, when n is sufficiently large, is $O(n^{k-\sigma})$.

To prove the rest of the theorem, I use a formula given by Boole,* but which is quite analogous to the Euler-Maclaurin sum-formula (1.3).

This can be established as follows† :—

If $F(x)$ be a continuous function and also its derivatives of the first $(2k_1 + 1)$ orders, (k_1 a positive integer, or zero), for $p \leqslant x \leqslant N$, then

$$(6.1) \qquad \frac{(-1)^\nu}{2}\{F(\nu+1)-F(\nu)\} = \int_\nu^{\nu+1} F'(x)\bar{\chi}_0(x)\,dx,$$

where
$$\bar{\chi}_0(x)\ddagger = 2\sum_0^{\infty} \frac{\sin(2\nu+1)\pi x}{(2\nu+1)\pi},$$

a periodic function, of which the value is equal to $\frac{1}{2}$ for

$$2\nu < x < 2\nu + 1$$

and equal to $-\frac{1}{2}$, for

$$2\nu + 1 < x < 2\nu + 2.$$

Now, putting

$$\nu = p,\ p+1,\ \ldots\ \ldots,\ N-1,$$

---

* Lindelöf, *loc. cit.*, pp. 78, *et seq.*

† Compare this method with that followed by Cramér (l. c., p. 17). in establishing the sum-formula (1.3).

‡ See Lindelöf, *loc. cit.*, p. 37.

successively, and adding the results, we get

$$(6.2)\qquad \sum_{p}^{N} (-1)^{\nu} F(\nu) = \tfrac{1}{2}\left\{(-1)^{p} F(p) + (-1)^{N} F(N)\right\} - \int_{p}^{N} \bar{\chi}_0(x) F'(x)\, dx.$$

This is the formula when F (x) possesses the first derivative only, *i.e.*, when $k_1 = 0$. When k_1 is an integer, we integrate the function $\bar{\chi}_0(x)\, F'(x)$ by parts, and make use of the properties* of

$$\bar{\chi}_{2k_1}(x) \equiv (-1)^{k_1} . 2 . (2k_1)! \sum_{\nu=0}^{\infty} \frac{\sin (2\nu+1)\pi x}{((2\nu+1)\pi)^{2k_1+1}}, \quad (k_1 = 1, 2, 3, \ldots),$$

and $$\bar{\chi}_{2k_1+1}(x) \equiv (-1)^{k_1+1} . 2 . (2k_1+1)! \sum_{\nu=0}^{\infty} \frac{\cos (2\nu+1)\pi x}{((2\nu+1)\pi)^{2k_1+2}},$$

$$(k_1 = 0, 1, 2, 3 \ldots),$$

then we get

$$(6.3)\qquad \sum_{p}^{N} (-1)^{\nu} F(\nu) = \tfrac{1}{2}\left((-1)^{p} F(p) + (-1)^{N} F(N)\right)$$

$$+ \sum_{\nu=1}^{k_1} (-1)^{\nu} \frac{B_\nu (2^{2\nu}-1)}{(2\nu)!} \left((-1)^{p} F^{(2\nu-1)}(p) - (-1)^{N} F^{(2\nu-1)}(N)\right) + R(k_1, N),$$

where B's have the usual significance, and

$$R(k_1, N) = \frac{-1}{(2k_1)!} \int_{p}^{N} \bar{\chi}_{2k_1}(x)\, F^{(2k_1+1)}(x)\, dx, \quad (k_1 = 1, 2, 3, \ldots).$$

Now the proof of the theorem is almost contained in that of Theorem I. Put F $(x) = \phi(x)\, e^{-sf(x)}$ in (6.3), and suppose that s is bounded, $\sigma > k$, and $N \to \infty$. Then

$$F^{r}(x) \to 0, \text{ as } x \to \infty, \qquad (r = 0, 1, 2, 3, \ldots).$$

---

* See Lindelöf, *loc. cit.*, p. 37.

Also the infinite integral for R is absolutely and uniformly convergent for $\sigma > k$, at the least, so that, when s is bounded and $\sigma > k$, the right side of (6.3) tends to a limit, as $N \to \infty$, and, therefore, the Dirichlet's series

$$\sum_{p}^{\infty} (-1)^{\nu} \phi(\nu) e^{-sf(\nu)}$$

converges for $\sigma > k$. Then we prove, as in the case of Theorem I, that the right side of (6.3), when $N \to \infty$, represents an integral function of s, so that the sum of our series is an integral function of s. As examples, we may quote the series

$$\sum_{\nu=1}^{\infty} (-1)^{\nu} \nu^{-s},$$

and

$$\sum_{8}^{\infty} (-1)^{n} n^{\beta} (\log n)^{\gamma} \exp\{-s(\log n + \log\log n)\}.$$

7. **Theorem VI.** *Let*

$$H(s) = \sum_{p}^{\infty} (-1)^{n} \phi(n) e^{-sf(n)}, \text{ where } \phi(x) \text{ and } f(x) \text{ satisfy}$$

the same conditions as in Theorem III, with this difference only that q is now an odd positive integer, instead of 'positive integer'; then the Dirichlet's series possesses at least a half-plane of convergence (though not necessarily, of absolute convergence), and the function $H(s)$ is an integral function of s.*

I suppress the proof of the theorem. which is quite similar to that of Theorem III, the only difference being that in one case we start with formula (1.3), and in the other with formula (6.3).

It may be remarked that if

$$\sigma \geqslant 1 + k' + \delta > 1 + k', \quad \delta > 0,$$

and $$\phi(x) = O(x^{-\gamma}), \qquad \text{where } \gamma > 0,$$

then q may be taken to be unity. In this case, formula (6.2) must be used.

* This is required by the form of the remainder R in the formula (6.3).

It follows from this theorem that the following series represent integral functions of s.

$$\sum_{1}^{\infty} (-1)^n e^{-s(\log n)^{\alpha}}, \quad 0 < \alpha < 1;$$

$$\sum_{3}^{\infty} (-1)^n e^{-s(\log \log n)^{\beta}}, \quad \beta > 0.$$

8. The following theorem effects the analytic continuation of the function represented by the series

$$\sum_{1}^{\infty} \frac{1}{n} \psi\left((\log n)^{\alpha}\right) e^{-s(\log n)^{\alpha}}, \quad 0 < \alpha < 1,$$

ψ satisfying certain conditions mentioned in the theorem, over a wider half-plane than that of absolute convergence. It also shows that there exist Dirichlet's series of types lower than $\log n$ which are summable (A)* over a wider half-plane than that of absolute convergence. In particular, the example given at the end of the theorem shows that we can construct series of types lower than $\log n$ which have a strip over which they are summable (A), but not summable (C), or summable by Riesz's typical means. I have failed to construct a Dirichlet's series of a type higher than $\log n$ having similar properties.

THEOREM VII. *Suppose $\psi(z)$ is an analytic function of $z = x + iy$, within and on the boundary of the region defined by $x \geqslant 0$ and $0 \leqslant y \leqslant p$, or $p \leqslant y \leqslant 0$, according as p is positive or negative, and satisfies throughout this region the relation*

$$\psi(z) = O(e^{Mx}), \text{ where } M \text{ is a constant.}$$

Suppose further that $\psi(x) = O(e^{kx})$,

and on the line $y = p$, $\psi(z) = O(e^{k_1 x})$, *where,* $k_1 < k$.

Then the series

$$H(s) = \sum_{1}^{\infty} \frac{1}{n} \psi\left((\log n)^{\alpha}\right) e^{-s(\log n)^{\alpha}}, \qquad 0 < \alpha < 1,$$

* See the next foot-note.

is absolutely convergent if $\sigma > k$, but the function $H(s)$ is regular for $\sigma > k_1$, and for a fixed value of σ comprised between k_1 and k,

$$H(s) = O\left(e^{(k(\sigma)+\varepsilon)|t|}\right),$$

where $k(\sigma) = \dfrac{k-\sigma}{k-k_1}|p|$, *($\varepsilon$, an arbitrarily small positive number), t tending to infinity with the same sign as p. On the other hand, if t and p have opposite signs,* $H(s) = O(|t|)$.

By Cauchy's fundamental formula*, it is easy to prove that

$$\psi'(x) = O\left(e^{(k+\varepsilon)x}\right).$$

So that, using the formula (1.3) (involving the first derivative only), we have

$$H(s) = G(s) + \int_1^\infty \frac{\psi((\log x)^\alpha)\, e^{-s(\log x)^\alpha}}{x}\, dx,$$

where $G(s)$ is an integral function of s and $= O(|s|)$, for a fixed value of σ.

Therefore $$H(s) = G(s) + J(s),$$

where

$$J(s) = \frac{1}{\alpha}\int_0^\infty \psi(x)\, e^{-sx}\, x^{(1/\alpha-1)}\, dx,$$

the integral being absolutely convergent if $\sigma > k$.

Now consider the integral

$$\int_R^{R+ip} \psi(z)\, e^{-sz}\, z^{(1/\alpha-1)}\, dz,$$

where $z = x + iy$, and $z^{(1/\alpha-1)} = e^{(1/\alpha-1)\log z}$,
where $\log z$ has its principal value.

Then this integral is of order

$$e^{|tp|-R(\sigma-M)} \cdot (R^2+p^2)^{\frac{1}{2}(1/\alpha-1)}$$

* Goursat, *loc. cit.*, § 33.

which $\to 0$, as $R \to \infty$, provided t is bounded, and $\sigma > M$. It follows, therefore, that if s lies in any finite portion of the half-plane $\sigma > M$,

$$(8.1) \qquad J(s) = \frac{1}{\alpha}\left(\int_0^{ip} + \int_{ip}^{ip+\infty} \psi(z)\, e^{-sz}\, z^{(1/\alpha - 1)}\, dz\right).$$

Now the first integral on the right side of (8.1) obviously represents an integral function of s, and if σ is fixed,

$$\left| \frac{1}{\alpha} \int_0^{ip} \psi(z)\, e^{-sz}\, z^{(1/\alpha - 1)}\, dz \right| < K\, \frac{e^{pt} - 1}{t}, \quad \text{(K, a constant).}$$

As regards the second integral on the right side of (8.1), we have on the path of integration,

$$\psi(z)\, e^{-sz}\, z^{(1/\alpha-1)} = O\left(e^{pt-(\sigma-k_1)x}\,(x^2+p^2)^{\frac{1}{2}(1/\alpha-1)}\right),$$

which shows that the integral is uniformly convergent in any finite part of the half-plane $\sigma > k_1$, and for a fixed value of $\sigma\ (> k_1)$, the integral is $O(e^{pt})$. It follows, therefore, that $J(s)$, and hence $H(s)$, is regular for $\sigma > k_1$,

Again, if t and p have the same signs, we have, for a fixed value of $\sigma > k_1$,

$$(8.2) \qquad H(s) = G(s) + O(e^{pt}) = O(e^{pt});$$

if they have opposite signs,

$$H(s) = O(|t|) + O\left(\frac{1}{t}\right) = O(|t|).$$

From the relation (8.2), we deduce, following Hardy,* that for a fixed value of σ comprised between k_1 and k,

$$H(s) = O\left(e^{\left(\frac{k-\sigma}{k-k_1}|p| + \varepsilon\right)|t|}\right), \qquad \varepsilon > 0,$$

t tending to infinity having the same sign as p.

The series is summable by Abel's method, or summable (A),† for $\sigma > k_1$, if

$$\frac{k-\sigma}{k-k_1}\cdot|p| < \frac{\pi}{2}.$$

---

* 'The application of Abel's method of summation to Dirichlet's series'—*Quarterly Journal*, Vol. 47. Theorems III and IV.

† The same reference as above, § 13.

As an application of the above theorem, we may consider the series

$$H(s) = \sum_{2}^{\infty} n^{Ai-1} (\log n)^{-\frac{1}{2}} e^{-s\sqrt{\log n}}, \qquad A > 0,$$

and following Hardy's method, we can show that

$$H(s) = O(|t|), \text{ when } t \to -\infty, \sigma < 0,$$

while $\quad H(s) \sim \mu\, 2\sqrt{\pi/A}\, e^{-\sigma t/2A}, \quad$ when $t \to \infty, \quad \sigma < 0,$

so that the index of $H(s)$ is $\frac{-\sigma}{2A}$, when $\sigma < 0$, and the series is summable (A) if $\sigma > -\pi A$, but not summable (A) if $\sigma < -\pi A$.

As the function $H(s)$ is not of finite order in the strip $-\pi A < \sigma < 0$, the series is not summable by Riesz's typical means* in this strip. It is easily proved that the function $H(s)$ is an integral function of s.

Part II.

9. There are Dirichlet's series such as

$$\sum_{1}^{\infty} e^{Ain} n^{-s}, \qquad (A \neq n\pi, \ n = 0, 1, 2, \ldots)$$

$$\sum_{1}^{\infty} e^{-s(\log n)^{\alpha}}, \quad \alpha > 1,$$

to the study of which the Euler-Maclaurin sum-formula (1.3), or its analogue, the Boole formula (6.3), can not be suitably applied. These, however, I find, can be studied by the use of Cauchy's theorem on residues.

Suppose we are given a Dirichlet's series

$$(9.1) \qquad \sum_{p}^{\infty} \phi(n)\, e^{-sf(n)}$$

such that $\phi(z)$ and $f(z)$ are analytic† functions of z in a certain region T enclosing the positive real axis from a certain point (say, $z = \beta$, $p - 1 < \beta < p$) onwards, and satisfy certain other conditions

---

* Hardy and Riesz, *loc. cit.* Theorem 45.

† While, in Part I, ϕ and f are continuous function of a real variable, (see Theorems I, III, IV, V and VI).

as to their order of magnitude, then applying Cauchy's theorem, we have the formula

$$(9.2) \qquad \sum_{p}^{n} \phi(\nu)\, e^{-sf(\nu)} = \int_S \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z}-1}\, dz,$$

where S is a closed contour lying entirely in T, and enclosing within it no other pole of $\left(\frac{1}{e^{2\pi i z}-1}\right)$ except $z = p, p+1, \ldots n$.

We shall hereafter be mainly concerned with the study of the analytic continuation of the Dirichlet's series (9.1) by means of the formula (9.2).

THEOREM VIII.—*If in the series (9.1)*

$\phi(n)$ and $f(n)$ satisfy the following conditions:—

(i) *$f(n) > 0$, for $n \geqslant p$;*

(ii) *$f(z)$ and $\phi(z)$ are analytic functions of $z = x+iy = \beta + \rho e^{i\psi}$, where $p-1 < \beta < p$ throughout a region T (which is supposed to include its frontier) enclosed by a curve C, whose equation is**

$$|y| = (\log(x+1-\beta))^{\gamma},$$

where γ is real and $\geqslant \alpha$ in (iii) (*a*);

(iii) *throughout the region T, and for sufficiently large values of* $|z|$,

(*a*) $$f(z) \sim (\log z)^{\alpha},$$

where α is real and $\geqslant 1$, and $(\log z)^{\alpha}$ has its principal value, and

(*b*) $$\phi(z) = O\left(|e^{kf(z)}| \,.\, e^{A|y|}\right),$$

where A and k are some real constants and $A < 2\pi$;

then $$H(s) = \sum_{p}^{\infty} \phi(n)\, e^{-sf(n)} = G(s) + J(s),$$

---

* The region T, as supposed here, is the minimum required for the truth of the Theorem. It is much smaller than an angular region, but greater than a rectangular one. The Theorem is, therefore, *a fortiori* true for an angular region.

where G (s) *is an integral function of* s, *and*

$$J(s) = \int_p^{\infty} \phi(x)\, e^{-sf(x)}\, dx.$$

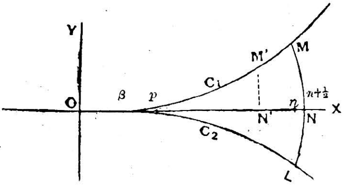
Both the Dirichlet's series and the corresponding integral are absolutely and uniformly convergent in the region

$$\sigma \geqslant 1 + k + \delta > 1 + k, \quad \text{or} \quad \sigma \geqslant k + \delta > k,$$

according as $\alpha = 1$, or > 1.

What we have to prove is that the difference between the functions represented by them is an integral function of s

Suppose then, for a moment, that s is real and $> 1 + k$. With



centre β and radius $R = n + \frac{1}{2} - \beta$, draw an arc of a circle cutting C_1 and C_2, the upper and lower parts of C, at M and L respectively.

Now, integrating the function $\phi(z)\, e^{-sf(z)} / (e^{2\pi i z} - 1)$, over the closed contour S, *i.e.*, βLNMβ, we have, by Cauchy's theorem,

$$(9.3) \qquad \sum_{p}^{n} \phi(\nu)\, e^{-sf(\nu)} = \int_S \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z} - 1}\, dz$$

$$= \int_{(\beta L)} + \int_{(LN)} + \int_{(NM)} - \int_{(\beta M)},$$

where $\int_{(PQ)}$ means that the integral is taken along the path PQ.

Now the sum of the last two integrals on the right-side of (9.3) is equivalent to

$$\int_{(NM)} \frac{\phi(z) e^{-sf(z)}}{e^{2\pi iz}-1} dz + \int_{(\beta M)} \phi(z) e^{-sf(z)} dz + \int_{(\beta M)} \frac{\phi(z) e^{-sf(z)}}{e^{-2\pi iz}-1} dz,$$

$$= \int_{(\beta N)} \phi(z) e^{-sf(z)} dz + \int_{(\beta M)} \frac{\phi(z) e^{-sf(z)}}{e^{-2\pi iz}-1} dz - \int_{(NM)} \frac{\phi(z) e^{-sf(z)}}{e^{-2\pi iz}-1} dz.$$

We, therefore, have

$$(9.4) \quad \sum_{p}^{n} \phi(\nu) e^{-sf(\nu)} = \int_{(\beta N)} \phi(z) e^{-sf(z)} dz$$

$$+ \int_{(\beta M)} \frac{\phi(z) e^{-sf(z)}}{e^{-2\pi iz}-1} dz + \int_{(\beta L)} \frac{\phi(z) e^{-sf(z)}}{e^{2\pi iz}-1} dz$$

$$+ \int_{(LN)} \frac{\phi(z) e^{-sf(z)}}{e^{2\pi iz}-1} dz + \int_{(NM)} \frac{\phi(z) e^{-sf(z)}}{1-e^{-2\pi iz}} dz.$$

Now we proceed to show that each of the last two integrals on the right side of (9.4) $\to 0$, as R, (and hence n) $\to \infty$, s being real and greater than $1+k$.

Putting $z = \beta + \rho e^{i\psi}$, we observe that

(9.5) $\left| \dfrac{1}{e^{2\pi iz}-1} \right|$ over the arc LN, and $\left| \dfrac{1}{e^{-2\pi iz}-1} \right|$ over the arc NM, each $< Ke^{-2\pi\rho |\sin\psi|}$, where K is some constant. Also, when $R \to \infty$, we have, over each of the arcs LN and NM, by virtue of conditions (iii) (*a*) and (*b*),

$$(9.6) \qquad |\phi(z) e^{-sf(z)}| = O(e^{A|y|} \, |e^{(k-s)f(z)}|)$$

$$= O(\exp. \{A|y| + (l^2+\tau^2)^{\frac{\alpha}{2}} (k-s) \cos \alpha\theta \}),$$

where

$$(9.7) \qquad l = \log \sqrt{\beta^2+\rho^2+2\beta\rho\cos\psi}, \quad \tau = \tan^{-1} \frac{\rho \sin\psi}{\rho\cos\psi+\beta},$$

$$\theta = \tan^{-1} \frac{\tau}{l}, \text{ and } \tau \text{ and } \theta$$

have their principal values, as well as the expression $(l+\tau i)^{\alpha}$. Therefore, the modulus of the integrand of each integral in question is

$O(\exp\{-(2\pi-A)\rho\,|\sin\psi| + (l^2+\tau^2)^{a/2}(k-s)\cos\theta\,a\}\rho)$,

for sufficiently large values of ρ.

Now integration being performed with respect to ψ, we observe that each integral in question $\to 0$, as $\rho \to \infty$, since $A < 2\pi$, $\sigma > 1+k$, and $a \geqslant 1$.

From (9.4), we therefore, have, if s is real and $> 1 + k$,

$$(9.8) \qquad H(s) = \sum_{p}^{\infty} \phi(\nu)\, e^{-sf(\nu)} = \int_{\beta}^{\infty} \phi(x)\, e^{-sf(x)}\, dx + I_1 + I_2,$$

where

$$I_1 \equiv \int_{\beta}^{\infty (C_1)} \frac{\phi(z)\, e^{-sf(z)}}{e^{-2\pi i z} - 1}\, dz, \quad I_2 \equiv \int_{\beta}^{\infty (C_2)} \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z} - 1}\, dz, \quad \int_{\beta}^{\infty (C)}$$

meaning that the integral is taken from β to ∞ along the arc C.

Now we proceed to show that each of the integrals I_1 and I_2 represents an integral function of s.

Suppose we first consider I_1. Put $z = x + iy$, where

$y = [\log(x + 1 - \beta)]^{\gamma}$, so that the integral I_1 can be transformed into one with respect to the independent variable x, which will move from β to ∞, as z moves from β to ∞ along C_1.

$$\text{Therefore,} \qquad I_1 = \int_{\beta}^{\infty} \frac{\phi(x+iy)\, e^{-sf(x+iy)}}{e^{-2\pi i(x+iy)} - 1} \left(1 + i\frac{dy}{dx}\right) dx.$$

Now to prove that I_1 is an integral function of s, it is sufficient to show that the integral

$$\int_{\beta}^{\infty} \left| \frac{\phi(x+iy)\, e^{-sf(x+iy)}}{e^{-2\pi i(x+iy)}} \left(1 + i\frac{dy}{dx}\right) \right| dx$$

is uniformly convergent throughout any finite domain of the values of s.

Now $\left|\left(1 + i\dfrac{dy}{dx}\right)\right| = \left|\sec\theta_1\right|$, where θ_1 is the angle

which the tangent to the curve C_1 makes with the positive real axis. It is also clear from the equation to the curve C_1 that $\sec\theta_1$ is

bounded.

$$\left| \frac{1}{e^{-2\pi i(x+iy)} - 1} \right| < Ke^{-2\pi |y|}$$

$$= Ke^{-2\pi (\log(x+1-\beta))^{\gamma}} \sim e^{-2\pi(\log x)^{\gamma}} . K,$$

where K is some constant. Also

$$| \phi(x+iy) e^{-sf(x+iy)} |$$

$$= O \left[\exp. \{ A \, | y | + \text{real part of } ((k - s) f(x+iy) \} \right],$$

for sufficiently large values of x.

Now the real part of $(k - s) f(x+iy)$ behaves like

$$(\log x)^{\alpha} \left(k - \sigma + \frac{t\alpha y}{x \log x} \right)$$

for sufficiently large values of x, so that, if t is bounded, the integrand of the last integral is

$$O(\exp. \{ -(2\pi - A)(\log x)^{\gamma} + (k-\sigma)(\log x)^{\alpha} \}).$$

Hence the integral is uniformly convergent throughout any finite domain of the values of s; for $A < 2\pi$, and $\gamma > \alpha$. I_1 is, therefore, an integral function of s.

Similarly, we can prove that I_2 represents an integral function of s. (9.8), therefore, becomes

(9.9) $$H(s) - \int_{\beta}^{\infty} \phi(x) e^{-sf(x)} dx = \text{an integral function of } s.$$

Now $$\int_{\beta}^{\infty} \phi(x) e^{-sf(x)} dx = \int_{\beta}^{p} + \int_{p}^{\infty} \phi(x) e^{-sf(x)} dx$$

where the first integral on the right side obviously represents an integral function of s. So that

$$H(s) - \int_{p}^{\infty} \phi(x) e^{-sf(x)} dx = G(s), \text{ an integral function of } s.$$

that is,

(9.10) $$H(s) - J(s) = G(s).$$

This equation has been obtained on the assumption that s is real and $> 1 + k$, but, as the right-side is an integral function of s, it persists

for all bounded values of s, according to the principle of analytic continuation.

This completes the proof of our theorem. The most obvious consequence of the relation (9.10) is that *the function $H(s)$ has no other finite singularities than those of $J(s)$, and vice versa.* The problem of studying the analytic continuation, situation of the singularities, and the behaviour near a singularity of the function $H(s)$ represented by the Dirichlet's series, is, therefore, resolved into one of studying the function $J(s)$.

10. THEOREM IX. *If the hypotheses of Theorem VIII hold, and further it is true that, for sufficiently large values of $|z|$.*

$$\phi(z) = O(|e^{kf(z)}| e^{-B|y|^{\lambda}}),$$

throughout T_1, the portion of T lying above the real axis and including its frontier, where B and λ are real positive numbers, and $\lambda\gamma > \alpha$, then the function $H(s)$ is an integral function of s.

We suppose, as before, that s is real and $> 1 + k$, in the first instance. From (9.3), we have

$$(10.1) \qquad \sum_{p}^{n} \phi(\nu)\, e^{-sf(\nu)} = \int_{(\beta L)} + \int_{(LN)} + \int_{(NM)} - \int_{(\beta M)} \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi z i} - 1}\, dz.$$

it was shown in the last section that

$$\left| \int_{(LN)} \right| \to 0, \text{ as } \rho \to \infty,$$

s being real and $> 1 + k$. Now we have to show that the same is true of the integral $\int_{(NM)}$.

Over NM and βM we have

$$(10.2) \qquad \left| \frac{1}{e^{2\pi i z} - 1} \right| < K_1, \qquad (K_1, \text{ a constant}).$$

Therefore, the modulus of the integrand of $\int_{(NM)}$ is

$$O\,[\rho.\exp.\{-B\,|y|^{\lambda} + (k - s)(t^2 + \tau^2)^{\alpha/2} \cos\theta\alpha\}],$$

$$= O\,[\rho.\exp.\{-B\,|y|^{\lambda} + (\log\rho)^{\alpha}(k - s)\}],$$

for sufficiently large values of ρ, where the quantities l, τ, and θ are the same as in (9.7).

Therefore

$$|\mathcal{J}_{NM}| \to 0, \text{ as } \rho \to \infty, \text{ since } B > 0,\ \alpha \geqslant 1,\ s > 1 + k.$$

We, therefore, have, if s is real and $> 1 + k$,

$$(10.3)\qquad H(s) = \sum_{p}^{\infty} \phi(\nu) e^{-sf(\nu)} = \int_{\beta}^{\infty (C_2)} \frac{\phi(z) e^{-sf(z)}}{e^{2\pi i z} - 1} dz$$

$$- \int_{\beta}^{\infty (O_1)} \frac{\phi(z) e^{-sf(z)}}{e^{2\pi i z} - 1} dz.$$

The first integral was shown, in the last section, to represent an integral function of s. Now we have to show that the same is true of the second integral. This is certainly the case if the integral

$$\int_{\beta}^{\infty} \left| \frac{\phi(x + iy)\, e^{-sf(x+iy)}}{e^{2\pi i (x + iy)} - 1} \left(1 + i\frac{dy}{dx}\right) \right| dx$$

where $\qquad z = x + iy$, and $y = (\log (x + 1 - \beta))^{\gamma}$,

is uniformly convergent throughout any finite domain of the values of s.

Making use of the inequality (10.2), and proceeding as in the last section, we have the integrand of the last integral to be

$$O\left(\exp\{-B(\log x)^{\gamma\lambda} + (k - \sigma)(\log x)^{\alpha}\}\right)$$

for sufficiently large values of x, t remaining bounded.

Now, as $B > 0$, $\lambda\gamma > \alpha$, the integral in question is uniformly convergent for all bounded values of s. Hence the integral

$$\int_{\beta}^{\infty (O_1)} \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z} - 1} dz.$$

represents an integral function of s

Therefore, the function $H(s)$ represented by the Dirichlet's series in the half-plane $\sigma \geqslant 1 + k + \delta > 1 + k$, or $\sigma > k + \delta > k$, according as $\alpha = 1$, or > 1, is continued analytically over the whole plane of s.

Corollary. If

$$H(s) = \sum_{p}^{\infty} (-1)^{s} \phi(n) e^{-sf(n)},$$

where $\phi(n)$ and $f(n)$ satisfy all the conditions of Theorem VIII, with this difference only that $A < \pi$; then $H(s)$ is an integral function of s.

For $$(-1)^z \phi(z) = e^{\pi i z} \phi(z)$$

which satisfies the additional condition of Theorem IX, besides satisfying the conditions of Theorem VIII.

Examples. The following interesting series all represent integral functions of s.

(i)* $\sum_{1}^{\infty} e^{Ain} n^{-s}$, (A $\neq$ a multiple of 2π);

(ii)† $\sum_{1}^{\infty} e^{Ain^{\alpha}} n^{-s}$, ($A \neq 0$, $0 < \alpha < 1$);

(iii) $\sum_{1}^{\infty} e^{Ain^{\gamma} - s(\log n)^{\alpha}}$ ($\alpha > 1$, $A \neq 0$, $0 < \gamma < 1$, or $\gamma = 1$, A $\neq$ a multiple of 2π).

11. By the way, I may prove the following theorem:

THEOREM X. *If*

(i) *$\phi(z)$ is an analytic function of z in a certain region T', such that, from and after a certain point $x_0 = x_0(\varepsilon)$, every point x of the positive real axis can be surrounded by a circle of radius εx, $0 < c \leqslant 1$, lying entirely within T', ε being an arbitrary positive number;*

(ii) *$\phi(z) = O(|z^k|)$, in the region T', k being a constant; then $\phi(x)$ satisfies conditions A and B of § 1, for $x \geqslant x_0$.*

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* S. Chapman: 'On non-integral orders of summability of series and integrals.' *Proc. Lond. Math. Soc*, Ser. 2, Vol. 9. 1910.

† G. H. Hardy: 'Theorems connected with Maclaurin's test for the convergence of series,' *Proc. Lond. Math. Soc*, 1910. Also H. Cramer, *loc. cit.*, p. 47.

As regard condition A, it at once follows from the fact that $\phi(z)$ is an analytic function of z in T′.

We have only to show that for sufficiently large values of x,

$$\phi^{\mu}(x) = O(x^{k-\mu c}), \quad (\mu = 0, 1, 2, 3, \ldots\ldots).$$

Now by Cauchy's formula, we have, if $x \geqslant x_0$,

$$\phi^{\mu}(x) = \frac{\mu!}{2\pi i}\int_c \frac{\phi(z)\,dz}{(z-x)^{\mu+1}},$$

the contour of integration being the circle of radius εx^c described round the point x as centre.

Now if x be sufficiently large,

$$|\phi^{\mu}(x)| = \frac{\mu!}{2\pi}\left|\int_C \frac{\phi(z)\,dz}{(z-x)^{\mu+1}}\right| \leqslant \frac{\mu!}{2\pi}\int_C \left|\frac{\phi(z)\,dz}{(z-x)^{\mu+1}}\right|,$$

$$= O\left[\frac{\mu!}{2\pi}\left(x + \varepsilon x^c\right)^k . \left(2\pi\right) . \frac{\varepsilon x^c}{(\varepsilon x^c)^{\mu+1}}\right],$$

$$= O\left[\frac{\mu!}{\varepsilon^{\mu}} . \left(x^{k-\mu c}\right)\left(1 + \frac{\varepsilon}{x^{1-c}}\right)^k\right],$$

$$= O(x^{k-\mu c}), \quad (\mu = 0, 1, 2, 3 \ldots).$$

It follows from this theorem that Theorem VIII as it relates to the series

$$\sum^{\infty} \phi(n)\, n^{-s},$$

where $\phi(z)$ satisfies the abovementioned hypotheses (i) and (ii), is included in Cramer's theorem (§ 1).

That Theorem VIII is not always included in Cramer's theorem is shown by the example

$$\sum_{1}^{\infty} e^{Ain}\, n^{-s}, \qquad (A \neq \text{a multiple of } 2\pi),$$

to which the former is applicable, but not the latter.

12. THEOREM XI. *Suppose in the series (9.1), ϕ and f satisfy the same conditions as in Theorem VIII, with these modifications only that $0 < \alpha < 1$, and $\gamma \geqslant 1$, the equality being admissible*

only when $(1 + A) < 2\pi$; *then the conclusions of Theorem III hold.*

The proof is practically the same as that of Theorem VIII. I shall briefly explain the difference in details.

Like the equation (9.4), we have, (see Fig. § 9),

$$(12.1) \quad \left\{ \sum_{p}^{n} \phi(\nu) e^{-sf(\nu)} - \int_{p}^{n} \phi(x) e^{-sf(x)} dx \right\}$$

$$= \left\{ \int_{\beta M} - \int_{(NM)} \frac{\phi(z) e^{-sf(z)}}{e^{-2\pi i z} - 1} dz \right\}$$

$$+ \left\{ \int_{\beta}^{p} + \int_{n}^{n+\frac{1}{2}} \phi(x) e^{-sf(x)} dx \right\}$$

$$+ \left\{ \int_{\beta L} + \int_{LN} \frac{\phi(z) e^{-sf(z)}}{e^{2\pi i z} - 1} dz \right\},$$

the integrals bracketed together having the same integrand.

Now suppose s lies in D and $n \to \infty$.

Then

$$\int_{n}^{n+\frac{1}{2}} \phi(x) e^{-sf(x)} dx = O\left[e^{(k-\sigma)(\log n)^{\alpha}}\right] \to 0.$$

Next, we wish to show that $\int_{NM}$ and $\int_{LN} \to 0$.

Now the modulus of the integrand of the first integral

$$= O\left(\left| e^{(k-s)(\log z)^{\alpha}} \right| e^{-(2\pi - A)\rho \sin\psi} . \rho \right)$$

$$= O\left[\rho . e^{-(2\pi - A)\rho \sin\psi}), \ \exp. \left\{ (k - \sigma)(\log \rho)^{\alpha} \cos\left(\frac{\alpha \bar{\psi}}{\log \rho}\right) + \right.\right.$$

$$\left.\left. |t| \left((\log \rho)^2 + \bar{\psi}^2\right)^{\frac{\alpha}{2}} . \frac{\alpha \bar{\psi}}{\log \rho} \right\}\right]$$

$$= O\left(\rho . e^{E} e^{-(2\pi - A)\rho \sin\psi}\right), \text{ say,}$$

where $\bar{\psi}$ is the upper limit of integration, and $\tan^{-1} \dfrac{\psi}{\log \rho}$ has its principal value.

Therefore

$$\left| \int_{NM} \right| = O\left(e^{E} \rho \int_0^{\bar{\psi}} e^{-(2\pi - A) 2\rho\psi/\pi} \, d\psi \right),^*$$

$$= O\left[e^{E} . \left(\frac{1 - e^{-(2\pi - A) 2\rho\bar{\psi}/\pi}}{(2\pi - A) 2/\pi} \right) \right],$$

which $\to 0$, as $\rho \to \infty$, s lying in D, and $A < 2\pi$.

Similarly, $\int_{LN} \to 0$, as $n \to \infty$, s lying in D.

Hence, if s lies in D, we have

$$(12.2) \qquad \lim_{n \to \infty} \left\{ \sum_{p}^{n} \phi(\nu)\, e^{-sf(\nu)} - \int_p^n \phi(x)\, e^{-sf(x)} dx \right\} =$$

$$\int_\beta^p \phi(x)\, e^{-sf(x)} dx + \int_\beta^{x\,(C_2)} \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z} - 1} dz + \int_\beta^{\infty\,(C_1)} \frac{\phi(z)\, e^{-sf(z)}}{e^{-\pi i z} - 1} dz.$$

Now we show, as in § 9, that the right side of (12.2) represents an integral function of s.

Here, while treating the last two integrals, it is necessary to suppose that $\gamma \geqslant 1$, the equality being admissible only when

$$(1 + A) < 2\pi.$$

Remarks.

(i) We have proved the theorem when $f(z) \sim (\log z)^{\alpha}$, $(0 < \alpha < 1)$, but its extension to such cases as $f(z) = (\log \log z)$, $(\log \log \log z)^{\beta}$, $(\beta > 0)$, etc. is obvious.

(ii) Now consider the series

$$H(s) = \sum_{1}^{\infty} e^{Ain - s(\log n)^{\alpha}}, \ (0 < \alpha < 1,\ A \neq 2\pi k,\ k = 0, 1, 2, 3 \ldots),$$

to which our Theorem III is not applicable.

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* For if

$$0 \leqslant \psi \leqslant \frac{\pi}{2}, \quad \sin \psi \geqslant \frac{2}{\pi} \psi.$$

This step was not necessary in the case of Theorem VIII; for α there was greater than or equal to unity.

By Theorem XI, we have

$$H(s) = G(s) + \int_1^{\infty} e^{Aix - s \log x^{\alpha}} dx,$$

the series and the integral both being convergent (conditionally), if $\sigma > 0$.

Now it is easy to show that the integral represents an integral function of s, so that $H(s)$ is an integral function of s.

Similarly, the series

$$\sum_3^{\infty} e^{Ain - s (\log \log n)^{\beta}}, \qquad (\beta > 0, \ A \neq 2\pi k, \ k = 0, 1, 2, \dots),$$

represents an integral function of s.

13. THEOREM XII. *The result of Theorem VIII is still true, if in condition (ii) the equation to the curve C is taken to be* $|y| = (x-\beta)^{\gamma}$, *where* γ *is real and* $\leqslant 1$, *and condition (iii) (a) is replaced by*

$$f(z) \sim z^{\alpha},$$

where α *is real and* $0 < \alpha < \gamma$,* z^{α} *having its principal value.*

The proof follows exactly the same lines of argument as that of Theorem VIII, and is, therefore, omitted.

14. THEOREM XIII. *Theorem IX is still true, if in its enunciation the number 'VIII' is replaced by 'XII'.*

The proof is similar to that of Theorem IX. There is a corollary to Theorem XIII similar to the one to Theorem IX.

Example.—The series

$$\sum_1^{\infty} e^{Ain^{\beta} - sn^{\alpha}},\dagger$$

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* I will consider the most interesting case $f(n) = n$ in another paper. From the detailed proof of the Theorem here, it will be evident why the extension to the case $\alpha > 1$ is not possible.

† For a particular case of this series, see G. H. Hardy: 'Example to illustrate a point in the Theory of Dirichlet's series.' *The Tohoku Math. Journal*, Vol. 8, No. 2, 1915.

$$\left(0 < \alpha < \beta \leqslant 1, \quad \begin{array}{c} A \neq 0,\ 0 < \beta < 1 \\ \text{or} \\ A \neq \text{a multiple of } 2\pi,\ \beta = 1 \end{array}\right),$$

represents an integral function of s.

15. In Theorems VIII, IX, XII and XIII, we may suppose

(i) *$\phi(z)$ is an analytic function of z throughout the region T, except for a finite number of isolated (polar and essential) singularities, which do not coincide with the poles of*

$$\frac{1}{e^{2\pi iz} - 1};$$

(ii) *there exists a positive number ρ_0 such that for $\rho \geqslant \rho_0$,*

(*a*) $\phi(z) = O(\,|e^{kf(z)}|\,)^*$, *throughout T_1*,

and (*b*) $\phi(z) = O(\,|e^{kf(z)}|\,e^{A|y|})$ *throughout T_2*,

where A and k are some real constants and $A < 2\pi$, T_1 and T_2 being the two portions of T lying respectively above and below the real axis and including their frontiers.

Now consider Theorem VIII, for example, under these conditions. If $\beta N \geqslant \rho_0$, in the figure we have only to subtract $2\pi i \times$ (the sum of the residues of the function

$$\frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi iz} - 1}$$

with respect to the singularities of $\phi(z)$) from the right-side of (9.3). Now this sum is evidently an integral function of s, say $G_1(s)$, so that

---

* This is to be replaced by

$$\phi(z) = O(\,|e^{kf(z)}|\, e^{-B|y|^{\lambda}}),$$

where B and λ are positive numbers and $\lambda\gamma > \alpha$, in the case of Theorems IX and XIII

equation (9.3) becomes

(15.1)
$$\sum_{p}^{n} \phi(\nu)\, e^{-sf(\nu)}$$

$$= -G_1(s) + \int_{(\beta L)} + \int_{(LN)} + \int_{(NM)} - \int_{(\beta M)} \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z} - 1}\, dz,$$

$$= -G_1(s) + \left\{ \int_{(\beta L)} + \int_{(LN)} + \int_{(NM)} \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z} - 1}\, dz \right\}$$

$$+ \int_{(\beta M)} \frac{\phi(z)\, e^{-sf(z)}}{e^{-2\pi i z} - 1}\, dz + \int_{(\beta M)} \phi(z)\, e^{-sf(z)}\, dz.$$

Now $\left| \int_{LN} \right| \to 0$, as $\rho \to \infty$, if s is real and $> 1 + k$, as in § 9. We have to show that the same is true of the integral

$$\int_{(NM)} \frac{\phi(z)\, e^{-sf(z)}}{e^{2\pi i z} - 1}\, dz.$$

Over the arc NM, $\left| \frac{1}{e^{2\pi i z - 1} - 1} \right| <$ a constant.

Therefore, the modulus of the integrand of $\int_{(NM)}$ is

$$O\left(\rho\, e^{(k-s)(\log \rho)^{\alpha}}\right), \text{ if } \rho \text{ be sufficiently large.}$$

Therefore $\left| \int_{(NM)} \right| \to 0$, as $\rho \to \infty$, if s is real and $> 1 + k$.

Now proceeding as in § 9, we easily have

(15.2)
$$H(s) = \sum_{p}^{\infty} \phi(\nu)\, e^{-sf(\nu)} = G(s) + \int_{\beta}^{\infty\,(0,\,)} \phi(z)\, e^{-sf(z)}\, dz,$$

where G (s) is an integral function of s.

Now with centre β and radius ρ_0 draw an arc of a circle cutting C_1 and the x-axis at M′ and N′ respectively. Then it is easy to see

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NOTES AND QUESTIONS.

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Notes and Questions.

The Contact Circle.*

The expressions obtained for the lengths of the tangents drawn from L, M, N, the middle points of the sides of a triangle ABC, to the contact circle of isogonal conjugate points S, S′ on the line $p\alpha + q\beta + r\gamma = 0$, passing through the symmedian point of the triangle, show that

$$t_1 : t_2 : t_3 = p : q : r. \quad \dots \quad \dots \quad (1)$$

Let Ω be the point of contact of the circle XYZ and the nine-point circle, and d the distance between the centres of these circles. From the theory of co-axal circles, we have,

$$t_1^2 : \mathrm{L}\Omega^2 = d : \frac{\mathrm{R}}{2}, \quad \dots \quad \dots \quad (2)$$

where R is the radius of the circle ABC.

From (2) and (3) we get,

$$\mathrm{L}\Omega : \mathrm{M}\Omega : \mathrm{N}\Omega = t_1 : t_2 : t_3 = p : q : r. \dots \quad \dots (3$$

Hence the normal co-ordinates of Ω referred to the medial triangle LMN, are

$$\frac{1}{p}, \frac{1}{q}, \frac{1}{r} \quad \dots \quad \dots \quad (4)$$

i.e., Ω *corresponds in the triangle LMN to the triangular pole of the line SS′ with respect to ABC.*

If the normal co-ordinates of Ω referred to the triangle LMN be given as $\sec\lambda$, $\sec\mu$, $\sec\nu$, we have

$$\mathrm{L}\Omega = \mathrm{R}\cos\lambda,\dagger$$

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* Continued from page *51*, No. 4, Vol XVII, *J.I.M S.*

† *Vide* Gallatly: *Modern Geometry of the Triangle* (second edition), page 26, § 38.

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and from (2)

$$t_1^2 = \frac{R^2 \cos^2 \lambda . d}{R/2} = 2Rd \cos^2 \lambda. \quad \dots \quad (5)$$

Also

$$4t_1^2 = p^2 \frac{-(bcp + caq + abr)^*}{pqr}. \quad \dots \quad (6)$$

It is simpler to use areal co-ordinates in expressing d, $\cos \lambda$, in terms of p, q, r. If the areal equation of the line SS′ referred to the triangle ABC be

$$px + qy + rz = 0,$$

the condition that it passes through the symmedian point (a^2, b^2, c^2) gives

$$a^2p + b^2q + c^2r = 0. \quad \dots \quad \dots \quad (7)$$

(6) takes now the form

$$4t_1^2 = \frac{-a^2p^2(p + q + r)}{pqr}. \quad \dots \quad \dots \quad (8)$$

$$= 8Rd \cos^2 \lambda,$$

and the relation connecting $p, q, r, \lambda, \mu, \nu$ become

$$\frac{\cos \lambda}{p \sin A} = \frac{\cos \mu}{q \sin B} = \frac{\cos \nu}{r \sin C}. \quad \dots \quad \dots \quad (9)$$

From (8) and (9) we get

$$2l = -R . \frac{\Sigma \sin B . \sin C \cos \lambda}{\cos \lambda \cos \mu \cos \nu}. \quad \dots \quad (10)$$

We know that λ, μ, ν, satisfy the relations

$$\sin A . \cos \lambda + \sin B . \cos \mu + \sin C \cos \nu = 0, \quad \dots \quad (11)$$

and $$\sin A \sin \lambda + \sin B \sin \mu + \sin C \sin \nu = 0, \quad \dots \quad (12)$$

which give $$\frac{\sin A}{\sin (\mu - \nu)} = \frac{\sin B}{\sin (\nu - \lambda)} = \frac{\sin C}{\sin (\lambda - \mu)}$$

and $$\frac{\cos A}{\cos (\mu - \nu)} = \frac{\cos B}{\cos (\nu - \lambda)} = \frac{\cos C}{\cos (\lambda - \mu)} = -1. \quad (13)$$

Putting each fraction in (9) equal to k and substituting $kq \sin B$, $kr \sin C$ for $\cos \mu$, $\cos \nu$ in the result

$$\cos (\mu - \nu) = -\cos A,$$

---

* *Vide J I.M.S.*, Vol. XVII, No. 4, page *51*, § (7),

given in (13), we obtain,

$$k^2 qr \sin B . \sin C + \surd \{ (1 - k^2 q^2 \sin^2 B)(1 - k^2 r^2 \sin^2 C) \} = - \cos A,$$

whence $k^2 (q^2 \sin^2 B + r^2 \sin^2 C + 2qr \sin B . \sin C \cos A) = \sin^2 A$ ...(14)

From (7) we have

$$p \sin^2 A + q \sin^2 B + r \sin^2 C = 0.$$

$$\therefore \quad q^2 \sin^2 B = q(- p \sin^2 A - r \sin^2 C),$$

and

$$r^2 \sin^2 C = r(- p \sin^2 A - q \sin^2 B).$$

On using the above values in (14), we find

$$k = \frac{1}{\surd (- qr - rp - pq)} \quad \ldots \quad \ldots \quad (15)$$

$$\therefore \quad \cos \lambda = \frac{p \sin A}{(-\Sigma qr)^{\frac{1}{2}}}, \quad \cos \mu = \frac{q \sin B}{(-\Sigma qr)^{\frac{1}{2}}}, \quad \cos = v \frac{r \sin C}{(-\Sigma qr)^{\frac{1}{2}}}. \quad (16)$$

We may also show that

$$\sin A . \sin \lambda = \frac{q \sin 2B - r \sin 2C}{(-\Sigma qr)^{\frac{1}{2}}},$$

$$\sin B . \sin \mu = \frac{r \sin 2C - p \sin 2A}{(-\Sigma qr)^{\frac{1}{2}}},$$

$$\sin C . \sin \nu = \frac{p \sin 2A - q \sin 2B}{(-\Sigma qr)^{\frac{1}{2}}}. \quad \ldots \quad \ldots \quad (17)$$

The relations (11) and (12) are at once verified by the values given in (16) and (17).

With the help of (10) and (16) we immediately find,

$$d = \frac{R}{2} . \left(p + q + r\right)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right) . \quad \ldots \quad (18)$$

Let ρ be the radius of the contact circle XYZ.

Then

$$\rho = \frac{R}{2} - d$$

$$= \frac{R}{2}\left\{1 - \left(p + q + r\right)\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right)\right\}$$

$$= - \frac{R}{2} \frac{(q + r)(r + p)(p + q)}{pqr}. \quad \ldots \quad (19)$$

Let $L'M'N'$ be the ortho-centric triangle of ABC, and t'_1, t'_2, t'_3 be the lengths of the tangents drawn from L', M', N' to the contact circle XYZ. The calculation of t'_1, t'_2, t'_3 may now be easily effected thus:—

In the cyclic quadrilateral MNLL′, the side MN is parallel to LL′. Therefore NL = L′M, and LM = NL′. In the cyclic quadrilateral $\Omega ML'N$, we have

$$\Omega L' . MN + \Omega M . NL' + \Omega N . L'M = 0$$

$$\therefore \quad \Omega L' . MN + \Omega M . LM + \Omega N . NL = 0.$$

Observing that ΩL, ΩM, ΩN are respectively equal to $R \cos \lambda$, $R \cos \mu$, $R \cos \nu$, the above relation may be written

$$\Omega L' . \sin A + R (\cos \mu \sin C + \cos \nu \sin B) = 0,$$

$$\therefore \quad \Omega L' = -R . \frac{\cos \mu \sin C + \cos \nu \sin B}{\sin A},$$

$$= -R . \frac{\sin B \sin C}{\sin A} . \frac{q + r}{(-qr - rp - pq)^{\frac{1}{2}}} \quad \ldots \quad (20)$$

on using the values given in (16).

From the relation, $t'^2_1 : \Omega L'^2 = d : \frac{R}{2}$,

we find $$t_1'^2 = \frac{R^2 . \sin^2 B . \sin^2 C}{\sin^2 A} . (q + r)^2 . \frac{-(p + q + r)}{pqr}, \quad (21)$$

by inserting the value of d given in (18).

Similarly $$t'^2_2 = \frac{R^2 \sin^2 C \sin^2 A}{\sin^2 B} . (r + p)^2 . \frac{-(p + q + r)}{pqr},$$

and $$t'^2_3 = \frac{R^2 \sin^2 A . \sin^2 B}{\sin^2 C} (p + q)^2 . \frac{-(p + q + r)}{pqr}.$$

Remembering that $$t_1^2 = R^2 p^2 \sin^2 A . \frac{-(p + q + r)}{pqr},$$

we get at once, $$\frac{t_1 t_2 t_3}{t'_1 t'_2 t'_3} = \pm \frac{pqr}{(q + r)(r + p)(p + q)}$$

$$= \frac{R}{2\rho}, \text{ from (19).} \quad \ldots \quad \ldots \quad (22)$$

Also $$\frac{\Omega L . \Omega M . \Omega N}{\Omega L' . \Omega M' . \Omega N'} = \frac{t_1 t_2 t_3}{t'_1 t'_2 t'_3} = \frac{R}{2\rho}. \quad \ldots \quad (23)$$

Observing that the latus-rectum of the parabola escribed to the triangle LMN with Ω as focus, is equal to

$$\frac{\Omega L \cdot \Omega M \cdot \Omega N}{R^2},$$

we may enunciate the result (23) as follows:—

If with any point on the nine-point circle as focus, parabolas are escribed to the medial and orthocentric triangles, their latera-recta are proportional to the radii of the nine-point circle and the contact circle touching the nine-point circle at the focus.

M. BHIMASENA RAO.

A test for groups of primes.*

1. By Wilson's theorem, if p be a prime

$$(p-1)!+1 \equiv 0 \pmod{p}$$

this condition being both necessary and sufficient. The object of this note is to point out that by repeated applications of Wilson's theorem we may obtain a test that a set of numbers, p, $p+r$, $p+s$, ... may all be prime numbers.

2. Let us find the conditions under which both p and $p+r$ are primes. Obviously r is even.

Since p and $p+r$ are primes, by Wilson's theorem

$$(p-1)!+1 \equiv 0 \pmod{p} \quad \dots \quad \dots \quad (1)$$

$$(p+r-1)!+1 \equiv 0 \pmod{p+r}. \quad \dots \quad (2)$$

Now, $$p+r-t \equiv -t \pmod{p+r}.$$

Hence substituting $-t$ for $p+r-t$, in (2) for all numbers from p to $p+r-1$, we get

$$(-1)^r\, r!\,(p-1)!+1 \equiv 0 \pmod{p+r}$$

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* This problem was kindly suggested to me by Prof. K. Ananda Rau.

or since r is even.

$$r!\,(p-1)! + 1 \equiv 0 \pmod{p+r}.$$

Let us write this in the form

$$(p-1)! + \frac{1}{r!} \equiv 0 \pmod{p+r} \quad \ldots \quad \ldots \quad (3)$$

noticing that $r!$ is prime to $(p+r)$.

Let $$\frac{(p-1)!+1}{p} + x \equiv 0 \pmod{p+r}.$$

Then $$(p-1)! + 1 + px \equiv 0 \pmod{p+r}$$

$$\therefore \quad (p-1)! + 1 - rx \equiv 0 \pmod{p+r}. \quad \ldots \quad (4)$$

From (3) and (4), we obtain

$$rx \equiv 1 - \frac{1}{r!} \pmod{p+r}$$

or $$x \equiv \frac{r!-1}{r.r!} \pmod{p+r}$$

$$\therefore \quad \frac{(p-1)!+1}{p} + \frac{r!-1}{r.r!} \equiv 0 \pmod{p+r}$$

$$\therefore \quad (p-1)! + 1 + p.\frac{r!-1}{r.r!} \equiv 0 \pmod{p \times \overline{p+r}}. \quad \ldots \quad (5)$$

So in order that p and p + r may both be primes, (5) is a necessary condition.

We shall now show that it is also sufficient.

For, let (5) be satisfied.

Write $$x \equiv \frac{r!-1}{r.r!} \pmod{p+r}$$

which means that

$$r.r!\,x = m\,(p+r) + r! - 1$$

where m is an integer.

Clearly we can choose m so that x is an integer also,

$$\therefore \quad (p-1)! + 1 + px \equiv 0 \pmod{p\,(p+r)}$$

$$\therefore \quad (p-1! + 1 \equiv 0 \pmod{p}.$$

whence it follows that p is a prime.

Again, from (5)

$$(p-1)!+1+p\,\frac{r!-1}{r.r!}\equiv 0 \pmod{p+r}$$

$$(p-1)!+1-\frac{r(r!-1)}{r.r!}\equiv 0 \pmod{p+r}$$

$$\therefore \qquad (p-1)!+\frac{1}{r!}\equiv 0 \pmod{p+r}$$

$$r!\,(p-1)!+1\equiv 0 \pmod{p+r}$$

Substituting $(p+r-t)$ for t for numbers in $r!$, this reduces to

$$(p+r-1)!+1\equiv 0 \pmod{p+r}.$$

Hence if (5) is satisfied, p and p + r are both primes.

Example.—Let $r = 2$;

then $$x\equiv\frac{2!-1}{2.2!} \pmod{p+2}$$

$$=\tfrac{1}{4} \quad \ldots \quad \ldots$$

$$\equiv-\frac{p+1}{4} \pmod{p+2}$$

$$\equiv\frac{p+3}{4} \pmod{p+2}.$$

Hence we obtain the result* given in Question 1437, *J.I.M.S.* Vol. XVI, Part II, Page *159*.

3. Let p, $p+s$, $p+r$ be all prime numbers where s and r are even and $r > s$.

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* 1437. (K. Ananda Rau):—If p is a prime number, prove that the necessary and sufficient condition that $p+2$ is also a prime number is

$$\frac{(p-1)!+1}{p}\equiv-\frac{p+3}{4} \pmod{p+2}$$

when p is of the form $(4m+1)$

and $$\frac{(p-1)!+1}{p}\equiv\frac{p+1}{4} \pmod{p+2}$$

when p is of the form $(4m+3)$.

Then from (5)

$$\frac{(p-1)!+1}{p}+\frac{r!-1}{r.r!}\equiv 0 \pmod{p+r}.$$

Let

$$\frac{\dfrac{(p-1)!+1}{p}+\dfrac{r!-1}{r.r!}}{p+r}+x\equiv 0 \pmod{p+s}$$

$$\therefore \quad \frac{(p-1)!+1}{p}+\frac{r!-1}{r.r!}+(p+r)x\equiv 0 \pmod{p+s}$$

$$\therefore \quad \frac{(p-1)!+1}{p}+\frac{r!-1}{r.r!}-(r-s)x\equiv 0 \pmod{p+s}. \quad (6)$$

But from (5)

$$\frac{(p-1)!+1}{p}+\frac{s!-1}{s.s!}\equiv 0 \pmod{p+s}. \quad (7)$$

∴ From (6) and (7), we get

$$x\equiv\left(\frac{r!-1}{r.r!}-\frac{s!-1}{s.s!}\right)\Big/(r-s) \pmod{p+s}.$$

So the necessary and sufficient condition that the above three numbers may be primes is

$$(p-1)!+1+\frac{p(r!-1)}{r.r!}+p(p+r)\left\{\left(\frac{r!-1}{r.r!}-\frac{s!-1}{s.s!}\right)\Big/(r-s)\right\}.$$
$$\equiv 0 \pmod{p(p+s)(p+r)}.$$

These results may be extended to groups of larger number of primes, but I have not been able to reduce the conditions for the general case to any simple form.

S. SIVASANKARANARAYANA PILLAI.

Books Received for Review.

Theory of Equations and the Complex Variable, by RAI CHARAN BISWAS. Chuckerverthy Chatterjee and Co., Ltd., Calcutta, 1926.

Solutions

Question 946.

(M. K. KEVALRAMANI) :—Show that there are forty points on the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 8x^2y^2,$$

such that if tangents be drawn from them to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

the points of contact have got their eccentric angles in the ratio of 1 : 2.

Remarks by A. Narasinga Rao.

The intersection of tangents at θ and 2θ to the ellipse is the point

$$x_1 = \frac{a\cos 3\theta/2}{\cos\theta/2} = \frac{a(1-3t^2)}{1+t^2}$$

$$y_1 = \frac{a\sin 3\theta/2}{\cos\theta/2} = \frac{a(3t-t^3)}{1+t^2}$$

where $t = \tan\theta/2$.

The locus of (x, y) is thus a cubic. If this point should also lie on the quartic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 8x^2y^2$$

it must coincide with any of the 4×3 intersections of the two curves.

Therefore there are only 12 such points.

Question 1010.

(K. J. SANJANA, M A.):—Solve the equation $x^3+y^3=a^3+b^3$ where a and b are given positive rational numbers and x and y are required to be rational and positive.

Examples :—

(i) $x^3 + y^3 = 19 = \left(\frac{5}{2}\right)^3 + \left(\frac{3}{2}\right)^3$

(ii) $x^3 + y^3 = 91 = 4^3 + 3^3$.

Solution and Remarks by the Proposer.

A solution to this question by Prof. N. B. Mitra appeared in the issue of this *Journal* for Aug. 1921 (Vol. XIII, page 156). By a slip Prof. Mitra has obtained $v = \frac{669}{196}$ in example (i) and $v = \frac{303}{37}$ in example (ii) and has thus inadvertently brought out incorrect results. The more general problem requires the expression of a number which is the sum of two rational cubes as the sum of two other cube numbers. The rational numbers may be positive or negative; but, in the numerical examples proposed, only positive numbers were required.

Let $N \equiv a^3 + b^3$ be resolvable into x^3+y^3. Assume $x = a - z$, so that

$$z^3 - 3az^2 + 3a^2z + b^3 = y^3. \quad \ldots \quad \ldots \quad (1)$$

Let y be set equal to $b + mz$; then from (1), as $z \neq 0$, we have

$$z^2(1 - m^3) - 3z(a + bm^2) + 3(a^2 - b^2m) = 0.$$

To avoid the quadratic irrationality, I further put $a^2 - b^2m = 0$ or $m = a^2/b^2$, and thus obtain

$$z = \frac{3(a + bm^2)}{1 - m^3} = \frac{3ab^3}{b^3 - a^3}.$$

As $x = a - z$, we get

$$x = \frac{a(a^3 + 2b^3)}{a^3 - b^3}. \quad \ldots \quad \ldots \quad (2)$$

and, as $y = b + mz$, we get

$$y = \frac{-b(2a^3 + b^3)}{a^3 - b^3}. \quad \ldots \quad \ldots \quad (3)$$

Thus finally

$$N \equiv a^3 + b^3 = a^3\left(\frac{a^3 + 2b^3}{a^3 - b^3}\right)^3 - b^3\left(\frac{2a^3 + b^3}{a^3 - b^3}\right)^3 \quad \ldots \quad (4)$$

which gives the general resolution.

This is easily verified, for, on transposing, we get

$$a^3\left\{\left(\frac{a^3+2b^3}{a^3-b^3}\right)^3-1\right\}=b^3\left\{\left(\frac{2a^3+b^3}{a^3-b^3}\right)^3+1\right\}$$

and on simplification, each is readily reduced to

$$9\,a^3b^3\,(a^6+a^3b^3+b^6)/(a^3-b^3)^3.$$

Prof. Mitra gets $2a^3-b^3$ in the numerator of y but this must be due to some slip.

In Example (1) $N \equiv 19 = \left(\frac{5}{2}\right)^3+\left(\frac{3}{2}\right)^3$.

Here by (2) $\quad x=\frac{895}{196}$ and by (3) $\quad y=-\frac{831}{196}$.

As y is negative we proceed to the next application of equation (4) and obtain—

$$x=\frac{895}{196}\left(\frac{895^3-2.831^3}{895^3+831^3}\right)=\frac{895\,(-430795007)}{196\,(1290773566)},$$

$$y=\frac{831}{196}\left(\frac{2.895^3-831^3}{895^3+831^3}\right)=\frac{831\,(859978559)}{196\,(1290773566)}.$$

It is thus seen that this resolution also is not in positive numbers and that it would be perfectly useless to apply the formulæ again to get positive numbers.

In Example (2), $N \equiv 91 = 4^3+3^3$. Here $x=\frac{472}{37}$, $y=-\frac{465}{37}$;

hence proceeding to a second application we get

$$x=\frac{472\,(472^3-2.465^3)}{472^3+465^3},\ y=\frac{465\,(2.472^3-465^3)}{472^3+465^3}.$$

It is seen that x is now negative and that it would be quite useless to apply the formulæ again to such unmanageable numbers.

Prof. Mitra deduces apparently positive solutions of both these examples, but this is due to the error referred to above.

The resolutions of 19 and 91 given in the examples were so framed as to lead to prodigiously lengthy positive solutions. Fairly manageable results can be obtained with other resolutions.

Thus for 19 we may take 3^3-2^3 or $\left(\frac{8}{3}\right)^3+\left(\frac{1}{3}\right)^3$. With the first set we obtain $19=(33^3+92^3)/35^3$, which is the simplest possible solution. Proceeding further we get

$$x=\frac{-33\,(1593313)}{35\,(742751)},\ y=\frac{92\,(850562)}{35\,(742751)}.$$

It will be found that the next solution gives positive values of x and y.

Again 91 is seen to be 6^3-5^3 or $\left(\frac{9}{2}\right)^3-\left(\frac{1}{2}\right)^3$. Taking the second set we get $x=\frac{9}{2}\left(\frac{9^3-2.1^3}{9^3+1^3}\right),\ y=\frac{1}{2}\left(\frac{2.9^3-1^3}{9^3+1^3}\right)$,

so that $$91=(6543^3+1457^3)/1460^3.$$

In general it is difficult to resolve any given number into the algebraic sum of two cubes; practically, however, if a number is so resolvable, we can obtain two or even three resolutions without much trouble. Writing down the cubic residues of the number we should be careful to note if when any residue is a cube, the corresponding quotient is also a cube. Thus for 19 two cubic residues 8 and -1 have the respective quotients 1 and 27; hence $19=3^3-2^3$ and also $=\left(\frac{8}{3}\right)^3+\left(\frac{1}{3}\right)^3$. For 91 the residues 64 ($=-27$, mod. 91) and 1 have the respective quotients 1 and 8.

In conclusion I may be permitted to express my deep sense of gratitude to Mr. Kaikobad F. Vakil for the patient and devoted care with which he has helped me in writing out this solution and in verifying its numerical results.

Question 1354.

(K. J. SANJANA):—P and Q are points on an ellipse whose eccentric angles are θ and ϕ; P′ and Q′ are the corresponding points on the nth pedal of the ellipse with respect to the focus S. Prove that

$$\left(\frac{SP}{SP'}\right)^{\frac{2}{n}}+\left(\frac{SQ}{SQ'}\right)^{\frac{2}{n}}=\frac{2-e^2(\cos^2\theta+\cos^2\phi)}{1-e^2}$$

Solution by M. V. Seshadri.

Let p_1 and r_1 be the perpendicular from the pole and the radius vector of the nth pedal, and p and r of the ellipse; we can regard the ellipse as

the *n*th negative pedal of the pedal curve whence

$$r_1 = \frac{r^{-n+1}}{p^{-n}} \quad i.e., \ SP' = (SP)^{1-n} p^n$$

$$\therefore \quad \frac{SP}{SP'} = \left(\frac{SP}{p}\right)^n \text{ or } \left(\frac{SP}{SP'}\right)^{\frac{2}{n}} = \frac{SP^2}{p^2}.$$

But the (p, r) equation of the ellipse with focus as pole is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1$$

$$\therefore \quad \frac{SP^2}{p^2} = \frac{2a \cdot SP - SP^2}{b^2}$$

For any point P, $SP = a + ex = a(1 + e\cos\phi)$.

$$2a \cdot SP - SP^2 = a^2 \left[2(1 + e\cos\theta) - 1 - 2e\cos\theta - e^2\cos^2\theta\right]$$
$$= a^2\left[1 - e^2\cos^2\theta\right]$$

$$\therefore \quad \left(\frac{SP}{SP'}\right)^{\frac{2}{n}} = \frac{a^2}{b^2}(1 - e^2\cos^2\theta) = \frac{1 - e^2\cos^2\theta}{1 - e^2}$$

Similarly $$\left(\frac{SQ}{SQ'}\right)^{\frac{2}{n}} = \frac{1 - e^2\cos^2\phi}{1 - e^2}.$$

$$\therefore \quad \left(\frac{SP}{SP'}\right)^{\frac{2}{n}} + \left(\frac{SQ}{SQ'}\right)^{\frac{2}{n}} = \frac{2 - e^2(\cos^2\theta + \cos^2\phi)}{1 - e^2}$$

Question 1385.

(V. TIRUVENKATACHARYA) :—Show that, if $|q| < 1$,

$$(1 - q^2)^2 (1 - q^4)(1 - q^6)^2 \ldots \times (1 + q^2)(1 + q^4)^2 (1 + q^6) \ldots\ldots$$
$$= 1 - q^2 - q^6 + q^{12} + q^{20} - q^{30} - q^{42} + + \ldots\ldots$$

Solution by S. D. Chowla.

It is a well-known result that if $|q| < 1$,

$$q_0 q_1^2 = 1 + q^2 + q^6 + q^{12} + q^{20} + \ldots\ldots + q^{n(n+1)} + \ldots\ldots$$

where $$q_0 = \prod_{n=1}^{\infty}(1 - q^{2n}), \quad q_1 = \prod_{n=1}^{\infty}(1 + q^{2n})^*$$

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* *Vide* Bromwich : *Infinite Series*, pp. 105-106.

For q write $\sqrt{-1}\,q$ in the result, and it becomes

$$\prod_{n=1}^{\infty}(1-q^{4n})(1+q^{4n-2})\times\prod_{n=1}^{\infty}(1+q^{4n})^2(1-q^{4n-2})^2$$
$$=1-q^2-q^6+q^{12}+q^{20}--++\ldots\ldots$$

i.e.,
$$\prod_{n=1}^{\infty}(1-q^{4n-2})^2(1-q^{4n})\times\prod_{n=1}^{\infty}(1+q^{4n-2})(1+q^{4n})^2$$
$$=1-q^2-q^6+q^{12}+q^{20}-q^{30}-q^{42}++\ldots\ldots$$

It is obvious that in the series on the right two positive terms are followed by two negative terms, which are again followed by two positive terms, and so on.

Question 1388

(V. RAMASWAMI AIYAR):—In a parabola, show that the isotomic conjugate of any diameter with respect to any self-conjugate triangle is parallel to the tangent at the extremity of that diameter.

Solution by T. S. Venkataraman.

Let the vertices of a triangle self-conjugate with respect to the parabola $y^2=4ax$ be

$$A(x_1\ y_1),\ B(x_2\ y_2),\ \text{and}\ C(x_3\ y_3).$$

Then

$$y_1y_2=2a(x_1+x_2);\quad y_1y_3=2a(x_1+x_3),\quad y_2y_3=2a(x_2+x_3)$$

and by subtraction $\quad y_1y_3-y_2y_3=2a(x_1-x_2).\quad\ldots\quad\ldots\quad(1)$

Let the diameter $y=y_4$ cut CB, CA in P, Q and let its isotomic conjugate cut CB, CA in P′, Q′ whose co-ordinates are $(x_5,\ y_5)$, $(x_6,\ y_6)$,

Then $\quad y_5=y_2-(y_4-y_3);\quad y_6=y_1-(y_4-y_3).\quad\ldots\quad(2)$

Also since $(x_5,\ y_5)$ is on the line $yy_1=2a(x+x_1)$ and $(x_6,\ y_6)$ on $yy_2=2a(x+x_2)$, it follows that

$$x_5=\frac{y_5y_1-2ax_1}{2a};\quad x_6=\frac{y_6y_2-2ax_2}{2a}.\quad\ldots\quad(3)$$

The slope of P′Q′ $=\dfrac{y_6-y_5}{x_6-x_5}=\dfrac{(y_1-y_2)\,2a}{(y_6y_2-y_5y_1)+2a(x_1-x_2)}$

which, by reduction with the help of (1) and (2) equals $2a/y_4$.

But $2a/y_4$ is also the slope of the tangent at the extremity of the diameter $y=y_4$. Hence that tangent and the isotomic conjugate are parallel.

Questions for Solution

Proposers of Questions are requested to send their own solutions along with their questions.

1488. (S. L. MALURKAR) :—Shew that

$$\cos n\theta = \frac{2^n \cdot n!}{(2n)!} \sin\theta \left(\frac{1}{\sin\theta} \frac{d}{d\theta} \right)^n (\sin\theta)^{2n-1},$$

where n is a positive integer.

1489. (S. SIVASANKARANARAYANA PILLAI) :—Examine the convergence of the following series:

$$\sum_{n=3}^{\infty} \frac{\phi(n)}{n^2 \log n \log \log n}$$

where $\phi(n)$ denotes the number of integers less than and prime to n.

1490. (S. SIVASANKARANARAYANA PILLAI) :—Examine the following statements :—

(i) Every prime factor of $m! + 1$ is congruent to 1 mod. m.

(ii) If there is no odd perfect number, then no number can be equal to $1\frac{1}{2}$ times the sum of its divisors.

(A perfect number is one which is equal to the sum of its divisors the number itself excluded).

1491. (S. D. CHOWLA) :—If m is odd, prove that

$$\sum_{s=1}^{s=m} e^{\pi i s^2/m} \sec(\pi s/m) = e^{\pi i/4} \sqrt{m} \left\{ e^{-\pi i/4m} - e^{-9\pi i/4m} + e^{-25\pi i/4m} - + \ldots \ldots \right\}$$

1492. (K. J. SANJANA):—Evaluate the definite integral

$$\int_0^\infty \frac{(a\cos^2 x + b\sin^2 x)\sin nx}{x}\,dx$$

where $n \geqslant 3$.

1493. (K. J. SANJANA):—A point P is taken at random within an acute-angled plane triangle; find the mean value of the sum of the sides of the pedal triangle of P with respect to the triangle.

Discuss the case when one of the angles of the given triangle is obtuse.

1494. (M. BHIMASENA RAO);—ABC is the diagonal triangle of a quadrangle LMNP. Show that the pedal circle of P with respect to the triangle LMN cuts the circle ABC at an angle equal to the complement of the sum of the angles PBC, PCA, PAB.

1495. (A. A. KRISHNASWAMI AYYANGAR):— A_1, B_1, C_1, D_1 are the circum-centres of the triangles BCD, CDA, DAB, ABC; A_2, B_2, C_2, D_2 are the circum-centres of the triangles $B_1C_1D_1$, $C_1D_1A_1$, $D_1A_1B_1$, $A_1B_1C_1$, and O, O_1, O_2 are the centres of the rectangular hyperbolas circumscribing the quadrangles ABCD, $A_1B_1C_1D_1$, $A_2B_2C_2D_2$. Show that the quadrangles ABCD, $A_2B_2C_2D_2$ are homothetic, the homothetic centre being the M-point defined in Question 1485. If MO_1 cuts the conic ABCDO at Q, R, show that O_1 is the middle point of QR and that

$$OO_2 \,.\, MO_2 = O_1O_2^2.$$

1496. (K. KUPPASWAMY):—Integrate the equations,

$$(1)\quad \frac{dx}{\log(y+z)} = \frac{dy}{\log(z+x)} = \frac{dz}{\log(x+y)}.$$

$$(2)\quad \frac{dx}{\cot(y+z)} = \frac{dy}{\cot(z+x)} = \frac{dz}{\cot(x+y)}.$$

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LIST OF BOOKS & JOURNALS RECEIVED IN THE LIBRARY

during the months of November and December 1927.

1 Acta Mathematica, **50,** 1 and 2.
2 American Journal of Mathematics, **49**, 3.
3 American Mathematical Monthly, **34,** 8 and 9.
4 Astrophysical Journal, **66**, 2.
5 Bulletin Academie des Sciences de L'Ukraine **2**, 2.
6 Bullitin of the American Mathematical Society, **33**, 5.
7 Bulletin des Sciences Mathematiques, Sept. and Oct., 1927.
8 Crelle's Journal, **158**, 2.
9 Deutsche Mathematiker Vereinigung, **36**, 9—12.
10 Mathematical Gazette, **13**, 190.
11 Mathematische Annalen, **98**, 304.
12 Messenger of Mathematics, **57**, 2 and 3, (2 copies).
13 Monthly Notices of the Royal Astronomical Society, **87**, 9
14 Nature, **119**, 2994—3008; **120**, 3009—3017.
15 Proceedings of the Cambridge Philosophical Society, **23**, 8.
16 Proceedings of the Physico-Mathematical Society of Japan, **5**, 4 and 5.
17 Philosophical Transactions of the Royal Society, **227**, 649.
18 Philosophical Magazine, **4**, 24 25 and 26.
19 Proceedings of the Royal Society, London, **116**, 775.
20 Recueil Mathématique de la Societe de la Plata, **33**, 4
21 Tohoku Mathematical Journal, **28**, 3, 4.
22 Transactions of the American Mathematical Society, **29**, 4 (2 copies).
23 Transactions of the Cambridge Philosophical Society, **23**, 11.

Books.

1 Encyklopädie der Mathematischen Wissenschaften, III (3), 7.
2 Bombay University Examination Papers, 1926.
3 The Summability of Single and Multiple Fourier's Series, Thesis, by *H. W. Bailey*.
4 Cubic Curves and Desmic Surfaces, Reprint, Trans, A.M.S., by *R. M. Mathews*.
5 Expansions in Terms of Certain Polynomials connected with the Gammer Function, 1925, Thesis, by *B. P. Hoover*.
6 Mapping by means of linear systems of curves invariant, under Cremona transformations, 1926, Thesis, by *T. L. Bennett*.
7 Note on Hyperorthogonants, by *Sir Thomas Muir* (Reprint).
8 The Theory of Hessians from 1883—1914, by *Sir Thomas Muir* (Reprint).

Numbers in black type refer to the volumes, and those in ordinary type to the numbers of the issues.

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